Abstract

This is a short note on isogonality, intended to exhibit the uses of isogonality in mathematical olympiads.
§ Notations

Here are some notations we’ll use in this note.

1. The abbreviation ‘wrt’ stands for ‘with respect to’.
2. \(I, O, H, G\) stand for the incenter, circumcenter, orthocenter and the centroid of \(\triangle ABC\) respectively.
3. \((XY), (XYZ)\) stand for the circle with \(XY\) as diameter and the circle through non-collinear \(X, Y, Z\) respectively.
4. \(P^*\) stands for the isogonal conjugate of a point \(P\).
5. \(P_{\infty \parallel MN}\) stands for the point at infinity on the line \(MN\).
6. \(\equiv\) stands for equality of cross-ratios due to perspectivity at \(P\).

§ Isogonai Lines

**Definition**

Two lines meeting at a point \(A\) are said to be isogonal with respect to an angle \(\angle BAC\) if they can be obtained by a reflection over the angle bisector of \(\angle BAC\).

Also if one line (not necessarily through \(A\)) can be obtained from the other via a reflection about the angle bisector and a homothety, then they are called antiparallel with respect to the angle.
• **Properties**

In the following we let $l$ and $m$ be isogonal lines with respect to an angle $\angle BAC$.

1. Let $P$ be a point on $l$ and the feet of perpendiculars from $P$ to $AB, AC$ be $P_c, P_b$. Then $P_bP_c \perp m$.

   *Proof.* Exercise.

2. (Symmedians Lemma) Let $M$ be the midpoint of $BC$ and let the tangents to the circumcircle meet at $X$. $AX \cap (ABC) = K$ and $AX \cap BC = J$. Then the following hold:
   
   - $AK$ is a symmedian in $\triangle ABC$.
   - $\frac{BJ}{JC} = \left(\frac{AB}{AC}\right)^2$.
   - $BCAK$ is a harmonic quadrilateral.
   - $ABK \sim AMC$.
   - $(AO)$ and $(BOC)$ meet on the midpoint of $AK$.
   - $BC$ is a symmedian in both $BAK$ and $CAK$.
   - $BC$ is one angle bisector of $\angle AMK$ and $MX$ is the other one.
   - Tangents to $(ABC)$ at $A$ and $K$ meet on $BC$.

   *Proof.* Exercise. One of the proofs of the first fact is in the examples. Try finding atleast 3 others.

3. Let the cevians $AD_1$ and $AD_2$ be isogonal wrt $BAC$. Then the circumcircles of $AD_1D_2$ and $ABC$ are tangent together.

   *Proof.* Exercise.

4. (Isogonal Line Lemma): Let $P$ and $Q$ be points on $l, m$ respectively. $BP$ intersects $CQ$ at $L_1$ and $BQ$ intersects $CP$ at $L_2$. Then $AL_1$ and $AL_2$ are isogonals wrt $\angle BAC$.

   *Proof.* Exercise. This also appears in St. Petersburg Mathematical Olympiad.

5. Let $P$ be a point on the perpendicular bisector of $BC$ and let $P'$ be its inverse in the circumcircle. Then $AP$ and $AP'$ are isogonal wrt $\angle BAC$.

   *Proof.* Follows from the Symmedians lemma.
\section*{\textsection Isogonal Conjugates}

\textbf{Definition}

Two points \( P \) and \( P^* \) are called isogonal conjugates if \( \overline{AP} \) and \( \overline{AP^*} \) are isogonal wrt \( \angle A \), \( \overline{BP} \) and \( \overline{BP^*} \) are isogonal wrt \( \angle B \) and \( \overline{CP} \) and \( \overline{CP^*} \) are isogonal wrt \( \angle C \).

\textit{Proof.} Usually proved using trig Ceva, but see the following for a synthetic proof.

\begin{itemize}
\item \textbf{Properties}
\end{itemize}

1. Let \( P \)'s reflections in \( BC, CA, AB \) be \( P_a, P_b, P_c \) respectively. Then the circumcenter of the triangle \( \triangle P_aP_bP_c \) is the isogonal conjugate of \( P \), henceforth denoted by \( P^* \).

\textit{Proof.} Let \( Q \) be the required circumcenter. Then from \( AQP_b \cong AQP_c \), \( \angle QAP_b = \angle QAP_c \), from which \( AQ \) and \( AP \) are isogonal wrt \( \angle BAC \). From this and other similar relations, \( P \) and \( Q \) are isogonal conjugates, as required.

2. If \( D_1 \) and \( D_2 \) are as in Property 3 of section on properties of isogonal lines, then \( \frac{BD_1}{CD_1} = \frac{BD_2}{CD_2} = \frac{AB^2}{AC^2} \).

\textit{Proof.} Exercise

3. The pedal triangles of \( P \) and its isogonal conjugate have the same circumcircle with the circumcenter being the midpoint of \( PP^* \).

\textit{Proof.} Homothety.

4. \( AP \cap BC = D \) and \( AP^* \cap (ABC) = D' \). Then \( AD.AD' = AB.AC \).

\textit{Proof.} Inversion

5. \( \angle BPC + \angle BP^*C = \angle A \)

\textit{Proof.} Angle chasing

6. The insimilicenter and the exsimilicenter of \((BPC)\) and \((BP^*C)\) lie on \((ABC)\).

\textit{Proof.} Exercise.

7. If the circumcenters of \((BPC)\) and \((BP^*C)\) are \( O_1 \) and \( O_2 \) then \( AO_1 \) and \( AO_2 \) are isogonal with respect to \( \angle BAC \).
Proof. Follows from the previous fact and Property 5 in the section on properties of isogonal lines.

8. Let $\mathcal{E}$ be a conic with foci $F_1$ and $F_2$, with the tangents $t_1$ and $t_2$ from a point $X$ meeting it at $X_1$ and $X_2$. Then the following hold true:

- $XX_1$ and $XX_2$ are isogonal with respect to $\angle X_1XX_2$.
- $XF_i$ bisects $\angle X_1F_iX_2$ where $i = 1$ or $2$.
- The normals and the tangents at $X_i$ bisect $\angle F_1X_iF_2$ where $i = 1$ or $2$.
- The reflection of $F_i$ over $t_i$ is collinear with the other focus and one tangency point.

Also for every pair of isogonal conjugates, a conic tangent to all three sides of $\triangle ABC$ and having them as the foci exists.

Proof. Exercise. Use a good characterisation of the tangent to the conic (sum and difference of distances).

Some useful isogonal conjugates

<table>
<thead>
<tr>
<th>Point</th>
<th>Isogonal Conjugate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthocenter</td>
<td>Circumcenter</td>
</tr>
<tr>
<td>Centroid</td>
<td>Point of concurrence of symmedians</td>
</tr>
<tr>
<td>Gergonne Point</td>
<td>Insimilicenter of the circumcircle and the incircle</td>
</tr>
<tr>
<td>Nagel Point</td>
<td>Exsimilicenter of the circumcircle and the incircle</td>
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<tr>
<td>In/Excenters</td>
<td>Themselves</td>
</tr>
<tr>
<td>Nine point center</td>
<td>Kosnita point</td>
</tr>
</tbody>
</table>

For more, see the Kimberling Encyclopedia of Triangle Centers.
§ Examples

1. Symmedians lemma

Proof. Let the reflection of $A$ in $M$ be $D$. Then $BD, BX$ are isogonal and $CD, CX$ are isogonal. Thus, $D$ and $X$ are isogonal conjugates, and so $AD$ and $AX$ are isogonal, as desired.

2. (IMO 2000 SL G3): Do there exist points $D, E, F$ on $BC, CA, AB$ of an acute triangle $\triangle ABC$ respectively such that $AD, BE, CF$ concur and $OD + DH = OE + EH = OF + FH = R$?

![Diagram](image)

Proof. Yes, there exist such points.
Consider the conic with foci $O, H$ and tangent to the sides of $\triangle ABC$. Since the reflections of $H$ over the sides lie on the circumcircle, for any point $X$ of the ellipse, $OX + XH = R$. Now Brianchon’s theorem finishes the problem.

3. (USAMO 2008 P2): Let the $A$–median of a triangle $ABC$ be $AM$, and let the perpendicular bisectors of $AB$ and $AC$ meet $AM$ at $D$ and $E$. $BD$ and $CE$ intersect at $F$. Prove that $A, F, O$ and the midpoints of $AB$ and $AC$ are concyclic.
Proof. Note that $\angle BFC = 2\angle A = \angle BOC$ and so $B, F, O, C$ lie on a circle. We need that the circle with $AO$ as diameter and $(BOC)$ meet at $F$, for which it suffices to show that $AF$ is a symmedian. Consider the isogonal conjugate of $F$, say $F^*$. By the angle conditions, both $(AF^*B)$ and $(AF^*C)$ are tangent to $BC$, and thus by radical axis, $AF^*$ is a median, which completes the proof.

The point $F^*$ is sometimes called the $A$–humpty point (Warning: this notation is not at all standard). We shall encounter this point later too.
4. (AoPS): Let $I$ be the incenter of $ABC$ and let $l$ be the line through $I$ and perpendicular to $AI$. The perpendicular to $AB$ through $B$ and that to $AC$ through $C$ meet $l$ at $E$ and $F$ respectively. The feet of perpendiculars from $E$ and $F$ to $l$ onto $BC$ are $M$ and $N$. Prove that $(AMN)$ and $(ABC)$ are tangent together.

**Proof.** Firstly note that if $I_b, I_c$ are the excenters opposite $B$ and $C$, then $B, A, E$ lie on the circle with $II_b$ as diameter and so $I_b, E, M$ are collinear by angle chasing. In fact, $I_bI_cFE$ is a rectangle.

If the external and the internal bisectors of $\angle BAC$ meet $BC$ at $D_2, D_1$ respectively, then,

$-1 = (B, C; D_1, D_2) \overset{I_b}{=} (I_c, I_b; A, D_2) \overset{P_{\infty \parallel AI}}{=} (N, M; D_1, D_2)$

and now since $AD_1 \perp AD_2$, they are the bisectors of $\angle MAN$, from which it follows that $AM$ and $AN$ are isogonal cevians, from which the conclusion follows. $\blacksquare$
§ Problems

The problems here are not sorted by difficulty; by easy we mean less interesting problems, by hard we mean interesting ones.

- Easy ones

1. Complete all the proofs left as exercises.

2. Prove that $IO = IH$ if and only if one of the angles of the triangle is $60^\circ$.

3. Prove that the isogonal of $AP$ wrt $\angle BPC$ and that of $AP^*$ wrt $\angle BP^*C$ are symmetric wrt $BC$. (In fact, they meet on the tangency point of some special conic with $BC$).

4. Prove the following facts about the $A$–Humpty point:
   - Circles tangent to $BC$ and passing through $A, B$ and $A, C$ respectively, the circle with $AH$ as diameter, the circumcircle of $BHC$, the $A$–median, and the circumcircle of the triangle formed by the midpoint of $AH$, the reflection of $O$ in $BC$ and the midpoint of $BC$ all pass through the $A$–Humpty point.
   - It is the inverse of the midpoint of $BC$ under the inversion with radius $\sqrt{AH \cdot AD}$ and center $A$ where $H$ is the orthocenter and $AD$ is an altitude in $ABC$.

5. Find the complex and the barycentric coordinates of the isogonal conjugate of an arbitrary point $P$, assuming that the circumcircle is the unit circle.

- Medium Problems

1. Let $O$ be the circumcenter of $ABC$ and let the tangents to the circumcircle at $A, B, C$ form a triangle $XYZ$, and let the orthic triangle of $ABC$ be $DEF$. Prove that the isogonal conjugate of $O$ wrt $DEF$ is the orthocenter of $XYZ$.

2. Let the points of intersection of $AP, BP, CP$ with $BC, CA, AB$ be $A', B', C'$ respectively. Then prove that if the isogonal conjugate of $P$ wrt $ABC$ is $Q$ then the reflections of $AQ, BQ, CQ$ in $B'C', C'A', A'B'$ respectively concur at a point.

3. Let $DEF$ be the pedal triangle of $P$ wrt $ABC$, and let the points $X, Y, Z$ be on $PD, PE, PF$ respectively such that $PD \cdot PX = PE \cdot PY = PF \cdot PZ$. Then prove that $AX, BY, CZ$ concur at a point whose isogonal conjugate is on the line $OP^*$.

4. Let $K$ be the symmedian point of $ABC$, the isogonal conjugate of $G$. Prove the following properties:
   - It is the unique point which is the centroid of its pedal triangle.
(b) The midpoint of the $A$–altitude, the midpoint of $BC$ and $K$ are collinear.

(c) (Lemoine circle): Lines parallel to the sides of the triangle through $K$ are drawn. Prove that the points of intersections of these parallels with the sides are concyclic.

(d) (Another Lemoine circle): Lines antiparallel to the sides of the triangle through $K$ are drawn, meeting the remaining sides in $U,V,W,X,Y,Z$. Prove that these points are concyclic.

(e) The symmedians meet $(ABC)$ at $K_a, K_b, K_c$. Prove that $K$ is the symmedian point of the triangle formed by these points too.

- **Harder tasks**

1. (EMMO 2016\footnote{This year’s test was named “Every Mathematician Must Outperform”}) Let $ABC$ be a triangle, the orthocenter of whose intouch triangle is $P$. The reflections of $P$ over the perpendicular bisectors of $BC, CA, AB$ are $X, Y, Z$, and the midpoints of $YZ, ZX, XY, BC, CA, AB$ are $D, E, F, A_1, B_1, C_1$. Prove that $A_1D, B_1E, C_1F$ concur at the radical center of the nine-point circles of $I_aBC, I_bCA, I_cAB$, where $I_a, I_b, I_c$ are the excenters of the triangle $ABC$, opposite to $A, B, C$ respectively.

2. (ELMO 2016, wording modified\footnote{This was “Elmo Lives Mostly Outside”}) Let $ABC$ be a triangle and let $DEF$ be the intouch triangle, $AD, BE, CF$ being cevians. $AI$ meets $DE, DF$ at $M, N$ respectively. $S$ and $T$ are points on $BC$ such that $\angle MSN = \angle MTN = 90^\circ$. Prove that:

   (a) $(AST)$ is tangent to the circumcircle of $ABC$.

   (b) $(AST)$ is tangent to the incircle of $ABC$.

   (c) $(AST)$ is tangent to the $A$–excircle of $ABC$.\footnote{This wasn’t part of the original problem, but looks good.}
§ Hints to harder tasks

1. For any general $P$, prove that the concurrency point is actually the complement of the isogonal conjugate of $P$, where the complement of any point is its image under the homothety $H(G, \frac{1}{2})$.

   – Then prove that the isogonal conjugate of the given point $P$ is the concurrency point of the Euler lines of $IBC, ICA, IAB$, known as the Schiffler point.

   – To finish, prove that the complement of the Schiffler point is the radical center of the given nine-point circles.

2. Use isogonal lines, inversion, etc, etc, etc.
§ References
