LESSON 3: THE PIGEONHOLE PRINCIPLE

In mathematics, the *pigeonhole principle* or *Dirichlet’s box principle* states that if \( n \) items are put into \( m \) pigeonholes with \( n > m \), then at least one pigeonhole must contain more than one item.

In the picture below there are 9 pigeonholes and 10 pigeons. At least one pigeonhole contains more than one pigeon.

1. Prove that among any three integers there are two whose sum is even.

*Solution:* The idea for this problem, as for most of the problems, is to choose some boxes (pigeonholes) and some items to be placed in the boxes such that at least one of the boxes will contain more than one item. In this case, the first box is the set of all even numbers, and the second box is the set of all the odd numbers. As any integer is either odd or even, and we have 3 integers in the problem, (at least) two of them will be in the same pigeonhole. For this reason, they will have the same parity and thus their sum will be even.

2. There are 22 pupils in a class. Prove that one can choose four of them that are born on the same day of the week.

*Solution:* Remember from the lecture that whenever we had \( n \) items (pigeons) and \( m \) boxes (pigeonholes), we divided \( n \) by \( m \), obtaining a quotient and a remainder. That is,

\[
n = m \times k + r, \quad 0 \leq r \leq m - 1.
\]

If \( r \neq 0 \), there is at least one box with at least \( k + 1 \) pigeons in it. We proved this by contradiction. Supposing there was a way of placing the pigeons such that there were at most \( k \) pigeons in each pigeonhole, the maximum number of the pigeons in the pigeonholes would have been \( m \times k \). Since \( n > m \times k \), not all the birds could have been housed in the pigeonhole, which contradicts our supposition.
In this problem, the pigeonholes are the days of the week and the pigeons are the children in the class. Since we have 22 pigeons, I mean, children, and only 7 pigeonholes, there will be at least 4 belonging to the same box, that is, born on the same day of the week. \((22 = 7 \times 3 + 1)\)

3. In a soccer tournament each of the ten teams plays against all the other teams exactly once. Prove that at every moment of the competition there are two teams having played the same number of matches.

**Solution:** Assume by contradiction that there exists a moment during the competition when any two teams have played a different number of matches. As there are 10 teams, each can play 0, 1, 2, ..., 9 matches during the tournament. Note that if a team has played 0 matches until that moment, it didn’t played with any team and if it had played 9 matches, then until that moment it had played with all the other teams.

The number of matches a team could have played played until a specific moment will represent our boxes and the teams will represent the pigeons. Since we assumed each team had played a different number of matches, because there are 10 teams and 10 boxes, each team will be placed in a different box. So there will be a team which has played 0 matches at that moment, a team which has played 1 match, ..., and a team which has played 9 matches. But here we get a contradiction, because, if a team has played 9 matches, then it has played with all the other teams, so there can’t exist a team which has played no matches at all. Therefore, our supposition was false, so at any moment of the competition there are two teams which have played the same number of matches.

4. Five equilateral triangles of equal size can cover an equilateral triangle \(T\) (the triangles may overlap and parts of them may also fall outside the triangle \(T\)). Prove that the triangle \(T\) can be covered by using only four of the five triangles.

**Solution:** Denote the side length of the equilateral triangle \(T\) with \(\ell\). We shall take as boxes our 5 equilateral triangles and as items the following 6 points: the 3 vertices of \(T\) and the 3 midpoints of the sides of \(T\). As the 5 triangles entirely cover \(T\), they also cover the 6 points mentioned, so, by Dirichlet’s principle, at least one of the 5 triangles covers (at least) two of the points. Hence, the whole segment formed by the points is contained in that triangle, so the length of its side is larger then the length of the segment. As the segment’s length is \(\frac{\ell}{2}\) (or \(\ell\)), the side of each of the equilateral triangles that cover \(T\) is larger than \(\frac{\ell}{2}\). It is easy to see that \(T\) can be completely covered with 4 equilateral triangles of side length \(\frac{\ell}{2}\). Then it can also be
covered with four of the five equilateral triangles given (their side is longer than \( \ell \)).

5. Let \( S \) be a subset of the set \( \{1, 2, 3, \ldots, 2n\} \). Assume that the equation \( x + y = 2n + 1 \) does not have solutions in \( S \) (i.e., if \( x, y \in S \), then \( x + y \neq 2n + 1 \)). What is the largest number of elements that \( S \) can have?

**Solution:** As the set \( S \) must not contain two numbers with sum \( 2n + 1 \), we shall look first at the pairs of the numbers from \( S \) whose sum is \( 2n + 1 \). These are

\[(1, 2n), (2, 2n - 1), \ldots, (n, n + 1).\]

These pairs which will represent the boxes. Also, \( S \) cannot contain both numbers from a pair, because the sum of these numbers is \( 2n + 1 \). So \( S \) can contain at most one of the elements of a pair. Since there are \( n \) pairs, \( S \) will have at most \( n \) elements. On the other hand, taking in \( S \) the first element of each pair gives an example of a set \( S \) with \( n \) elements that fulfills the conditions from the statement \((S = \{1, 2, \ldots, n\})\). The fact that there are such sets \( S \) with \( n \) elements but not with more than that, shows that the desired maximum is \( n \).

6. Let \( n \geq 1 \) be a positive integer. If \( a_1, \ldots, a_n \) are positive integers, prove that it is possible to paint some of these numbers green in such a way that the sum of green numbers is divisible by \( n \). [Engel]

**Solution:** Whenever you have a combinatorics problem that asks you to prove that at least one number from a set of numbers is divisible by \( n \), you should think of considering as pigeonholes the possible remainders of the division by \( n \). That is, the first box represents all the integers that are divisible by \( n \), the second one all the integers that give remainder 1 when divided with \( n \), and so on until the last box, which represents all the integers that give remainder \( n - 1 \) when divided with \( n \). When we say that a number belongs to the box \( i \), we mean that the number gives remainder \( i - 1 \) when divided with \( n \).

For this reason, we need to choose \( n \) sums of some numbers from the set \( \{a_1, a_2, \ldots, a_n\} \) such that the difference of any two of them is still a sum of some \( a_i \)'s. So we are going to choose the following sums: \( A_1 = a_1, A_2 = a_1 + a_2, A_3 = a_1 + a_2 + a_3, \ldots, A_n = a_1 + a_2 + \cdots + a_n \). Since there are \( n \) sums and there are exactly \( n \) possible remainders when dividing by \( n \), we have 2 cases:

- Each \( A_i \) gives a different remainder when divided by \( n \). This means there is an \( A_i \) that gives remainder 0, so it is divisible by \( n \), hence our problem is solved. (We paint the numbers \( a_1, a_2, \ldots, a_i \) green.)
There are two $A_i$’s that give the same remainder when divided by $n$. Suppose we have $A_i$ and $A_j$ that are equal modulo $n$. Suppose $i < j$. We have that $A_j - A_i$ is divisible by $n$. So $a_1 + a_2 + \ldots + a_i + a_{i+1} + \ldots + a_j - (a_1 + a_2 + \ldots + a_i)$ is divisible by $n$, which means that $a_{i+1} + a_{i+2} + \ldots + a_j$ is divisible by $n$. So we obtained some numbers from the initial set of $a_i$’s with the sum divisible by $n$. (In this case we paint green the numbers $a_{i+1}, a_{i+2}, \ldots, a_j$.)

7. Prove that among any six people one can choose either three such that each two of them know each other, or three people such that in this group of three nobody knows anybody. [Kőrshak Competition 1947, Putnam 1953]

Solution: Denote the six persons by $A$, $B$, $C$, $D$, $E$ and $F$. Person $A$ has a relationship with each of the remaining people, it can either know one or not know one. Because there are five persons left and two kinds of relationships, there will be at least three people with whom he has the same relationship. Consider the case in which there are 3 people $A$ knows. We look at the relationships between $(the)$ 3 acquaintances of $A$. If two of them know each other, together with $A$ they form a group in which everybody knows everybody, and we are done. If in this group nobody knows anybody else, again, we are done. The case where $A$ has three people he does not know is similar: if in this group two people do not know each other, together with $A$ they form a group where nobody knows anybody else, and we are done. If we can not find two such persons then this means that in that group everybody knows everybody and, again, we are done.

8. On a blackboard there are 11 positive integers. Show that one can choose some (maybe all) of these numbers and place ”+” and ”−” in between, such that the result is divisible by 2011.

Solution 1: Denote by $S$ the set of the 11 integers. The first thing to observe is that 2011 is smaller than $2^{11} = 2048$, which is the total number of subsets of $S$. We will assign to each subset of $S$ the sum of its elements. As there are 2048 sums of the subsets and only 2011 possible remainders at the division by 2011, there will be at least two subsets whose sums will be equal modulo 2011. Thus, their difference will be divisible by 2011. But their difference is exactly what we need, since it consists of some of the initial 11 integers with a + or a − between them.

Solution 2: Consider the following $2^{11} = 2048$ numbers: $\pm a_1 \pm a_2 \pm \ldots \pm a_{11}$. Since there are more numbers then possible remainders at the division by 2011, at least two of these numbers will give the same remainder. Then their difference is a multiple of 2011. Their difference will be a sum of the numbers $a_i$ having coefficients −2 or 0 or 2. Since 2011 and 2 are relatively prime, half of the number we obtained is still a multiple of 2011 and it is also a sum of some of the 11 numbers.
with + or − in between.

9. Let \( S \) be a set of 27 distinct positive odd integers less than 100. Prove that there exist two elements in \( S \) with sum 102.

Solution: Consider the following pairs: \((3, 99), (5, 97), \ldots, (49, 53)\). Observe that the sum of the numbers in all these pairs is 102. We have 24 such pairs, and the numbers 1 and 51 are not included in any of these pairs. Even if 1 and 51 are both among the 27 chosen numbers, we still need to choose at least 25 of the 48 numbers appearing in the 24 pairs. By the Pigeonhole Principle, at least one of these pairs must be included in \( S \), so there are two elements in \( S \) that sum to 102.

10. Among the first 2012 positive integers, \(1006+n\) integers \((n \leq 1006)\) are colored green. Prove that there are \(2n\) green integers whose sum is divisible by 2013.

Solution: Consider the following pairs: \((1, 2012), (2, 2011), \ldots, (1006, 1007)\). The sum of the numbers in each pair is 2013. Because we have 1006 pairs and 1006 + \(n\) green numbers, by the Pigeonhole principle, at least \(n\) pairs will consist only of green numbers. Therefore, the total sum of the numbers in these \(n\) pairs is \(2013 \times n\) and we get \(2n\) green integers with their sum divisible by 2013.

11. There are 298 students at the university. Prove that 100 of these students can be painted green such that the difference of the ID numbers of every two green students is divisible by 3.

Solution: We shall take the ID numbers of the students modulo 3, i.e., we shall only look at their remainder when divided by 3. Since we have 298 ID numbers and 3 possible remainders, by Dirichlet’s principle, at least 100 of the numbers will give the same remainder when divided by 3 \((298 = 3 \times 99 + 1)\). Then the difference of any two of them is a multiple of 3. We choose 100 of the students possessing these ID numbers to be painted green.

12. Does there exist a positive integer \(n\) such that the decimal representation of \(3^n\) ends in “00001”?

Solution 1: Yes, actually there are infinitely many such numbers. Let us consider the following numbers: \(3^1, 3^2, \ldots, 3^n\), with \(m > 10^5\). Because we have \(m \geq 10^5 + 1\) numbers and only \(10^5\) possible remainders upon division by \(10^5\), at least two of the numbers will give the same remainder. Suppose these numbers are \(3^i\) and \(3^j\), with \(i < j\). Then \(10^5 \mid 3^j - 3^i\), which means that \(10^5 \mid 3^i(3^{j-i} - 1)\). Since \((10, 3) = 1\), we have \(10^5 \mid 3^{j-i} - 1\), so \(3^{j-i} - 1\) ends in 00000. Then \(3^{j-i}\) ends in 00001.
Solution 2: If we denote by $\varphi$ Euler’s totient function, we have $3^{\varphi(100000)} \equiv 1 \pmod{100000}$, which means that $3^{\varphi(100000)}$ ends in 00001.

13. Does there exist a positive integer $n$ such that $3^n$ has 2011 consecutive zeroes in its decimal representation?

Solution: The same argument as for the previous problem, only with $10^{2012}$ instead of $10^5$ allows us to prove that there are (infinitely many) powers of 3 ending in 000...001, that is, powers of 3 containing 2011 consecutive zeros.

14. Five points are chosen in a $3 \times 4$ rectangle. Prove that there are two of them whose distance is at most $\frac{5}{2}$.

Solution: Let us divide the rectangle into four identical parts by two lines, a horizontal one and a vertical one passing through the midpoints of the sides. Since we have four small rectangles $2 \times \frac{3}{2}$ and five points, by the Pigeonhole principle, two of the points will belong to the same rectangle. However, the largest distance that can be attained in a $2 \times \frac{3}{2}$ rectangle is the size of its diagonal, which is $\frac{5}{2}$, by Pythagora’s Theorem, so we have found two points at distance at most $\frac{5}{2}$.

15. Six points are chosen in a $3 \times 4$ rectangle. Prove that there are two of them whose distance is at most $\sqrt{5}$. [Engel]

Solution: Let’s observe that $\sqrt{5}$ is the size of the diagonal of a $2 \times 1$ rectangle. We shall call these rectangles dominoes. If we have two points in the same domino, the distance between them will be less than the size of its diagonal, which is $\sqrt{5}$. Therefore, no matter how we tile the $3 \times 4$ rectangle with dominoes, we have to have exactly one point in each domino (since there are six points and six dominoes), otherwise we are done. Let’s denote the columns of the $3 \times 4$ rectangle by $A, B, C, D$ (from left to right) and its rows by 1, 2, 3 (from top to bottom). By placing a domino in $B2$ and $C2$, we must have exactly one of the six points in one of the two unit squares. Without loss of generality suppose the point is in $C2$. By looking at the domino placed on $D2$ and $C2$, we can see that no point can be placed in $D2$, since otherwise we shall have two points at distance less or equal to $\sqrt{5}$. By looking at the domino placed on $D2$ and $D1$, since there is no point in $D2$ and there has to be a point in this domino, there will be a point in $D1$. By the same argument, there will be a point in $B3$. Now, the circles centered at any point of $B3$ and $D1$, respectively, and of radius $\sqrt{5}$ completely cover the square $C2$, so the point from $C2$ will be at a smaller distance to either the point from $D1$ or the
point from $B3$ than $\sqrt{5}$.

16. Let $S$ be any subset of ten natural numbers between 1 and 100 (inclusive). Prove that there are two disjoint non-empty subsets of $S$ whose elements add to the same amount. (IMO 1972)

Solution: The maximum sum that can be obtained from adding ten or less numbers between 1 and 100 is $100 + 99 + 98 + 97 + 96 + 95 + 94 + 93 + 92 + 91 = 955$ and the minimum sum is 1. All the other sums will be in between these two. The number of sums we can obtain from the 10 elements of $S$ is $2^{10} - 1$ (one for each non-empty subset), which is obviously larger than 955. Therefore, by the Pigeonhole principle, (at least) two of the subsets will have the same sum of their elements. If they share some common elements, we can just remove them from both sets, and we are left with two disjoint subsets of $S$ whose elements add up to the same value.

Practice problems

1. A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 black socks. A youngster selects socks one at the time from the drawer but is unable to see the color of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? (A pair of socks is two socks of the same color. No sock may be counted in more than one pair.) [AHSME 1986]

   (A) 21  (B) 23  (C) 24  (D) 30  (E) 50

Solution: We prove that the desired minimum is 23. Consider the following pigeonholes: the box of the red socks, the box of the green socks, the box of the blue socks and the box of the black socks. Now, you can pair up the socks in each box; in the end you will have at most one unpaired sock in each box. Consider 23 socks. Suppose at most 18 of them contribute to making pairs (at most 9 pairs). Then you would have 5 unpaired socks. You only have 4 boxes, hence one box should contain at least two unpaired socks. Pair them up and you will have at most 3 unpaired socks, hence at least 10 complete pairs. Therefore 23 socks are enough. Next we prove that 22 or less socks are not necessarily enough. Consider 9 complete pairs and 4 other socks, one of each color. We can not make the 10-th pair. Examples for less than 22 socks are obtained by removing socks from the distribution given for 22.

2. Consider a set $A$ with 10 elements. Prove that the intersection of any 513 subsets of $A$ is empty.
**Solution 1:** Consider the $2^{10} = 1024$ subsets of $A$ and pair them up into 512 pairs $(X, A\setminus X)$, with $X \subseteq A$. Whenever you take 513 subsets of $A$, by Dirichlet’s principle, you will have to choose at least 2 sets from the same pair. But $X \cap (A \setminus X) = \emptyset$, so the whole intersection of the 513 sets will be empty. Therefore, no matter how we choose 513 subsets of $A$, they will have an empty intersection.

**Solution 2:** Suppose the intersection is not empty. Then it contains at least one element $a$. This means that $a$ is an element of 513 of the subsets of $A$. But $a$ is an element of only $2^9 = 512$ of the subsets of $A$, which leads to a contradiction.

3. In every square of a $7 \times 7$ board there is a bug. At one moment all the bugs crawl into a neighboring square, that is into a square having a common side with the initial square of the bug. Prove that at least one square will be free of bugs.

**Solution:** Let us color the board as a chessboard, with black and white alternately, such that the upper left corner is black. As the board is 7 by 7, we shall have 25 black squares and 24 white squares. Whenever a bug crawls into a neighboring square (it cannot go diagonally), it switches the color of the square it moves into. When all the bugs move, they move from 25 black squares into only 24 white squares, so two of them will move on the same square. Therefore, at least one of the squares will remain free.

4. Prove that 2011 has a multiple whose decimal representation contains only the digit 1. [Engel]

**Solution:** Let us take the following numbers: 1, 11, 111, ..., $\underbrace{11\ldots1}_{2012}$. Each of them gives a remainder when divided by 2011, and since there are 2012 numbers, two of them give the same remainder. If two of these numbers are congruent modulo 2011, then their difference is divisible by 2011. Let us say these numbers are $a = \underbrace{11\ldots1}_{i}$ and $b = \underbrace{11\ldots1}_{j}$, with $i < j$. Since $2011 \mid b - a$, we get $2011 \mid \underbrace{11\ldots1}_{j-i}$. But $(2011, 10) = 1$, so $2011 \mid \underbrace{11\ldots1}_{j-i}$. Therefore, we found a multiple of 2011 written only with the digit 1.

5. Prove that among any 100 integers one can select 15 integers such that the difference between any two selected numbers is divisible by 7.

**Solution:** We shall look again at the remainder upon division by 7. Since there are 7 possible remainders and 100 numbers, by Dirichlet’s principle we have at least
15 numbers which give the same remainder, because \(100 = 7 \times 14 + 2\). Therefore, the difference of any two of these 15 numbers will be divisible by 7.

6. Given any 5 points on a sphere, show that some four of them must lie on a closed hemisphere.

**Solution:** Let us take two of the points which are not collinear with the center of the sphere (since there are 5 points, obviously there are not all collinear with the center). These two points together with the center of the sphere form a plane that cuts the sphere into two hemispheres. Since we are left with 3 points, two of them will belong to the same hemisphere. Hence, the first two chosen points and the last two points belong to the same hemisphere.

7. A \(4 \times 4\) square is completely covered with 13 overlapping \(2 \times 1\) dominoes. Prove that one can remove one of them such that the square remains completely covered.

**Solution:** Suppose that each domino was essential, i.e. removing it would leave an uncovered unit square. This means that for each domino there is a unit square only covered by that domino. These simply covered unit squares are distinct for different dominoes, hence there are at least 13 unit squares only covered by a single domino. But the 13 dominoes cover together 26 unit squares, so the remaining 3 unit squares are covered in total 13 times. By the Pigeonhole Principle, one of these unit squares will be covered at least 5 times. As there are only 4 distinct ways (at most) to place a domino in order to cover a given unit square, 2 of these 5 dominoes must be in the same position. But then be could remove one of them, which contradicts our initial assumption.